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Fluctuational distribution function of solitons in the nonlinear Schrödinger system

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We derive the Fokker-Planck equation for the soliton amplitude distribution function in the damped nonlinear Schrödinger system under the influence of thermal fluctuations. The regular solutions are obtained in terms of the Kummer functions. Physical applications are discussed.

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I. INTRODUCTION

Fluctuations are well known to play an important role in various soliton-bearing systems (see, e.g., the reviews [1] and [2]). In the presence of thermal fluctuations and of the dissipative damping related to them by the fluctuation-dissipation theorem (FDT) [3], solitons can be regarded as particles constituting a rarefied gas brought into contact with a thermostat [4]. Following this idea one can calculate various statistical characteristics of the soliton gas. In Ref. [4], the mean kinetic energy of a topological soliton (kink) and some other quantities have been found within the framework of the sine-Gordon (SG) model incorporating the lossy term and the randomly fluctuating force:

$$\phi_{tt} - \phi_{xx} + \sin\phi = -\alpha\phi_t + f(x,t), \tag{1}$$

where $f(x,t)$ is a Gaussian random function defined by the correlations

$$\langle f(x,t)\rangle = 0, \tag{2a}$$

$$\langle f(x,t)f(x',t')\rangle = \epsilon^2\delta(x-x')\delta(t-t'). \tag{2b}$$

The correlation amplitude ϵ^2 in Eq. (2b) is related to the damping constant α in Eq. (1) according to the FDT [4]:

$$\epsilon^2 = 2kT\alpha, \tag{3}$$

where T is the temperature and k is the Boltzmann constant.

In this work, the aim is to consider statistical properties of the gas of envelope solitons, which are the small-amplitude limit of the breather solutions in the SG model. Confining attention to small amplitudes, we will obtain from Eq. (1) a nonlinear Schrödinger (NLS) model with dissipative and fluctuation terms. The statistical properties of the soliton gas are determined by the Fokker-Planck (FP) equation for the distribution function of solitons. For the case opposite that considered here, i.e., for the small-frequency SG breathers, the FP equations have been derived and analyzed in Ref. [5]. The solutions to that FP equation (Ref. [5]) described the decay of the small-frequency breathers into kink-antikink pairs under the action of the random force. For the case considered here, i.e., for the envelope solitons, the FP equations corresponding to the physically important parameters of the soliton amplitude η and the velocity v are rather complicated in the general case. However, we will demonstrate that integrating the full distribution function $Q(\eta, v)$ over the velocity, one can derive the FP equation for the amplitude distribution function $P(\eta)$. This equation can be investigated in a fully analytical form. We demonstrate that, whatever the initial distribution, its asymptotic form (at $t\to\infty$) takes a very simple exponential form. Thus the steady-state mean value of the amplitude proves to be quite universal, depending only on properties of the nonlinear medium and on the temperature. We also demonstrate that the FP equation naturally conserves the total number of solitons. Thus, we expect that the time asymptotic mean density of solitons is not universal, being determined by the initial state of the system. At the end of the paper, we briefly discuss the situation for the case of parametric (multiplicative) noise in Eq. (1). Our preliminary inference is that, unlike additive noise, the parametric noise does not support any nontrivial distribution over the amplitudes of the envelope solitons: The integral that determines the total number of solitons strongly diverges at the infinitesimal amplitudes. In the concluding part of the paper, we also discuss physical applications of the model considered. We conclude that, among the possible applications, the most interesting may be the analysis of thermally induced noise in a nonlinear optical medium.

II. FOKKER-PLANCK EQUATION

To consider the small-amplitude envelope solitons, one starts from the well-known procedure of taking the small-amplitude limit of Eq. (1) [6]. One inserts into Eq. (1),

$$\phi(x,t) = u(x,t)e^{-it} + \text{c.c.}, \tag{4}$$

where the complex envelope $u(x,t)$ is assumed to be a slowly varying function of t compared with $\exp(-it)$. Then, after an evident rescaling, the effective NLS equation can be written as follows:

$$iu_t + u_{xx} + 2|u|^2u = -i\alpha u + f(x,t)e^{-it}. \tag{5}$$

Keeping only the slow time variation in the equation for $u(x,t)$, the main contribution from the Fourier expansion of the random function $f(x,t)$ will come for frequencies ω close to $\omega_0 = 1$. Thus one expands the random function

$f(x,t)$ as a Fourier integral and then keeps only the frequencies ω sufficiently close to $\omega_0 = 1$, which combined with e^{it} on the right-hand side (rhs) of Eq. (5) would comply with the condition of the slow evolution. The omission of the nonresonant terms could be done later, but this, however, would complicate the calculations and hide the main points of this paper. As will be explained below the formal representation in (5) is consistent for the derivation of the FP equation.

The unperturbed NLS soliton has the form

$$u_{\rm sol}(x,t) = 2i\eta\,{\rm sech}\{2\eta[x-\xi(t)]\} \times \exp\left(\frac{i}{2}vx - \frac{i}{4}v^2t + i\theta(t)\right), \qquad (6)$$

where η and v are its amplitude and velocity. The center-of-mass coordinate $\xi(t)$ and the phase $\theta(t)$ evolve simply with time as follows:

$$\frac{d\xi}{dt} = v, \quad \frac{d\theta}{dt} = 4\eta^2. \qquad (7)$$

The unperturbed NLS equation has an infinite set of integrals of motion, the simplest of which are the wave action (also called "number of particles")

$$N \equiv \int_{-\infty}^{+\infty} |u(x)|^2 dx \qquad (8)$$

and the momentum

$$K \equiv i\int_{-\infty}^{+\infty} u_x^*(x)u(x)dx. \qquad (9)$$

For the soliton in (6),

$$N_{\rm sol} = 4\eta, \quad K_{\rm sol} = 2\eta v. \qquad (10)$$

In the presence of the perturbations in the NLS model in Eq. (5), one can derive evolution equations for the amplitude and the velocity, using the so-called balance equations for the wave action and momentum [2]. A simple algebra yields

$$\frac{d\eta}{dt} = -2\alpha\eta - \frac{i}{4}\int_{-\infty}^{+\infty}[u^*(x)f(x,t)e^{it} - {\rm c.c.}]dx, \qquad (11)$$

$$\frac{dv}{dt} = \frac{1}{2\eta}\int_{-\infty}^{+\infty}[u_x^*(x)f(x,t)e^{it} + {\rm c.c.}]dx + \frac{iv}{4\eta}\int_{-\infty}^{+\infty}[u^*(x)f(x,t)e^{it} - {\rm c.c.}]dx. \qquad (12)$$

To derive the corresponding FP equation, we consider Eqs. (11) and (12), together with Eqs. (7), as the general Langevin equations [7]. The perturbing terms in Eq. (5) induce small corrections to Eqs. (7); however, they may be neglected in the lowest approximation of the perturbation theory [5]. The FP equation for the distribution function $Q(\eta,v,\xi,\theta;t)$ corresponding to these Langevin equations can be derived, making use of the correlations in (2), and it takes a pretty involved form. To simplify the equation, one can integrate it over v, ξ, and θ, in an attempt to derive the FP equation for the amplitude distribution function

$$P(\eta,t) \equiv \int_{-\infty}^{+\infty} dv \int_{-\infty}^{+\infty} d\xi \int_0^{2\pi} d\theta\, Q(\eta,v,\xi,\theta,t). \qquad (13)$$

The integration eliminates all the terms of the full FP equation that are full derivatives with respect to any of the variables v, ξ, and θ. The important fact, which can be noticed without detailed calculations, is that the coefficients of the surviving terms of the equation depend only on η, but not on the integrated variables. For the final derivation, it is necessary to mention that, when taking correlations of the rhs's of Eqs. (11) and (12) with the use of Eqs. (2), one will obtain terms of two different types. There will be terms that contain $\exp(\pm it)$ and rapidly oscillating ones containing $\exp(\pm 2it)$. We drop the latter terms, and one can readily check that this exactly corresponds to using from the very beginning the approximation in (5), when only the slowly oscillating Fourier components of the term $f(x,t)e^{it}$ are retained.

Finally, the FP equation for the integrated distribution function (13) takes the form

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial\eta}[(\tfrac{1}{4}\epsilon^2 - 2\alpha\eta)P] + \frac{\partial^2}{\partial\eta^2}(\tfrac{1}{4}\epsilon^2\eta P). \qquad (14)$$

Recall that ϵ^2 is the correlation amplitude in Eq. (2b) and measures the temperature of our system according to Eq. (3).

The FP equation (14) satisfies, as it should, the conservation of the total probability: it can be represented in the form $\frac{\partial P}{\partial t} = -\frac{\partial}{\partial\eta}J$, where the current is given by

$$J = -2\alpha\eta P - \frac{1}{4}\epsilon^2\eta\frac{dP}{d\eta}. \qquad (15)$$

An important property of the current (15) is the fact that it identically vanishes at $\eta = 0$, provided that P and dP/dt are finite at this point. This implies that there is no flux of probability across the point $\eta = 0$, and the total number of the solitons, which is proportional to $\int_0^\infty P(\eta)d\eta$, is conserved. This actually means that we should not deal with the problem of the creation of new solitons, which would exist for a nonzero value of $J(\eta)$ at $\eta = 0$. It is well known that the creation of new solitons by a perturbation is a very hard problem [2].

III. SOLUTION OF THE FOKKER-PLANCK EQUATION

A. Regular solutions

Equation (14) falls into a general class of FP equations considered in Refs. [12] and [13]. Below we will develop an analysis of Eq. (13) in a form slightly different from that put forward in these papers. However, all the results that we will obtain can also be obtained by means of the technique of Refs. [12] and [13]. First, we will look for eigenmodes of Eq. (14) in the form

$$P(\eta,t) = e^{-\gamma t}R(\eta) \qquad (16)$$

with a decay rate γ. Insertion of Eq. (16) into Eq. (14)

brings us to the equation

$$y\frac{d^2R}{dy^2} + (1+y)\frac{dR}{dy} + (1+\Gamma)R = 0, \tag{17}$$

where $y \equiv 8\alpha\epsilon^{-2}\eta$ and $\Gamma \equiv \gamma/2\alpha$. Equation (17) is exactly the confluent hypergeometric equation, if the sign of y in (17) is changed ($y \to -y$), with the standard parameters $a \equiv 1+\Gamma$ and $b \equiv 1$ [8]. The solutions that are finite at $y = 0$ and decay at $y \to \infty$ are given by the Kummer function $M(1+\Gamma, 1, -y)$ [8]. The limit of this function at $\Gamma = 0$ is just $\exp(-y)$. Note that the condition that provides for the vanishing of $M(1+\Gamma, 1, -y)$ faster than y^{-1} is $\Gamma \geq 0$, which assures that the total probability converges. This is also the condition that assures that the solutions in (16) do not grow as $t \to \infty$.

Thus, if one deals with an arbitrary initial distribution function $P_0(\eta)$, which is finite at $\eta = 0$ and vanishes at $\eta \to \infty$, it is natural to represent it as

$$P_0(\eta) = \int_0^\infty M(1+8\alpha\epsilon^{-2}\gamma, 1, -8\alpha\epsilon^{-2}\eta)\Pi(\gamma)d\gamma + \frac{8\alpha}{\epsilon^2}e^{-8\alpha\epsilon^{-2}\eta}, \tag{18}$$

with a kernel $\Pi(\Gamma)$. In the representation of Eq. (18), we took into account the fact that, for any positive Γ,

$$\int_0^\infty M(1+\Gamma, 1, -y)dy \equiv 0,$$

so that all the total probability, $\int_0^\infty P_0(\eta)d\eta \equiv 1$, is borne by the additional term corresponding to $\gamma = 0$. Then, looking at Eq. (16), one immediately concludes that, at $t > 0$, the distribution function becomes

$$P(\eta, t) = \int_0^\infty M\left(1+\frac{8\alpha}{\epsilon^2}, 1, -\frac{8\alpha\eta}{\epsilon^2}\right)\Pi(\gamma)e^{-\gamma t}d\gamma + \frac{8\alpha}{\epsilon^2}e^{-\frac{8\alpha\eta}{\epsilon^2}}. \tag{19}$$

Thus, at $t \to \infty$ the first term in Eq. (19) dies out, and the distribution function takes the static form

$$P_{\rm st}(\eta) = \frac{8\alpha}{\epsilon^2}e^{-\frac{8\alpha\eta}{\epsilon^2}}. \tag{20}$$

This static solution can as well be obtained directly from the class of static exponential solutions for FP equations of a more general form found in Ref. [12]. By substituting for ϵ from Eq. (3) in (20) the argument of the exponential becomes $-N_{\rm sol}/2kT$, where $N_{\rm sol}$ is clearly the energy for the case of the electromagnetic field, as seen from Eq. (8), where $|\, u(x)\,|^2$ is the energy density of the electric field. Thus Eq. (20) is a typical Maxwellian distribution.

B. Singular solutions

This case corresponds to an initial distribution $P_0(\eta)$, which has an integrable singularity at $\eta = 0$; e.g., $P_0(\eta) \sim \ln(1/\eta)$. The confluent hypergeometric equation (17) has a second linearly independent solution, which is usually designated as $U(1+\Gamma, 1, -y)$ [8]. With Γ positive these functions vanish at $y \to +\infty$ faster than y^{-1}, and they all have a logarithmic singularity at $y = 0$, so that the corresponding total probability converges. Therefore, one can consider an initial distribution with an integrable singularity at $\eta = 0$, represented by Eq. (19) with M replaced by an arbitrary linear combination of M and U. However, the additional term in Eq. (15) corresponding to $\Gamma = 0$ can only be taken in the purely exponential form. The second independent solution at $\gamma = 0$ is

$$e^{-y}\int_{y_0}^{y} e^{y'}\frac{dy'}{y'}, \tag{21}$$

which goes like y^{-1} at $y \to \infty$ and, thus, does not provide the convergence of the total probability.

The generalized solution with the logarithmic singularity keeps it at any finite value of t, but the singularity vanishes assymptotically at $t \to \infty$. It is important to notice that the logarithmic singularity yields a *nonzero* value of the current at $\eta = 0$ [see Eq. (15)]. Thus, the generalized solution implies a change in the total number of solitons. Using Eq. (16), it is easy to see that the change is finite, provided that the integral $\int_0^\infty \tilde{\Pi}(\gamma)\gamma^{-1}d\gamma$ converges, $\tilde{\Pi}(\gamma)$ being the kernel corresponding to the contribution of the U functions to the generalized solution. The real meaning of the generalized solution remains unclear because the production of new solitons of the fluctuating force is not properly comprised of the underlying Langevin equations (11) and (12); actually, this process should be analyzed separately (being a difficult problem, as was mentioned above). If one tries to use the function in (21) as another static distribution, it gives rise to a nonzero value of the current in (15) at $\eta = 0$. The corresponding value of $J(0)$ is positive, and this permanent influx of new solitons is the physical explanation for the divergence of the total probability corresponding to the solution (21). The existence of the singular stationary solutions of the FP equations is known in other contexts, and sometimes they can be given a physical meaning, depending on a particular situation; see, e.g., Ref. [13].

IV. PHYSICAL APPLICATIONS AND GENERALIZATIONS

The NLS equation, considered as the small-amplitude version of the full SG model, appears in a number of physical systems. Two common examples are long Josephson junctions [4] and charge-density-wave conductors [6]. In the former system, the small-amplitude envelope solitons are not a physically important entity. In the latter system, they may play a significant role, as they should determine the response of the conductor to an external ac electric signal. However, it seems more promising to apply the results obtained to physical systems in which the cubic nonlinearity plays a fundamental role. Important examples of such nonlinear media are known in optics, viz., optical fibers and planar wave guides with the Kerr nonlinearity (see, e.g., Ref. [9]). In this case, the quantity (8) has the meaning of the electromagnetic field energy, and thus Eq. (20) gives nothing but the distribution of solitons over energy in the state of thermodynamical equilibrium. In particular, the mean energy of the soliton is

$$\bar{E} \equiv 4\,\bar{\eta} = \frac{\epsilon^2}{2\alpha}, \tag{22}$$

the bar standing for the averaging over the static distribution. These results, applied to the Kerr media, may be of real interest, as they yield the statistical characteristics of the equilibrium electromagnetic noise (to be more accurate, of the soliton component of the "full" noise). On the other hand, it is necessary to remember that, while we are able to determine physical quantities like the mean soliton's energy, given in (22), we cannot find the density of solitons. In the framework of the developed approach, the density is determined by the initial condition. So, the FP equation yields only a part of the full statistical information.

The FP equation for the envelope solitons can be generalized in various directions. First of all, one can consider the dissipation terms of a more general type. If one starts from the SG model of the long Josephson junction, it is well known that the dissipative term written in Eq. (10) takes into account the shunt losses, while the so-called surface losses give rise to the additional term βu_{txx} [10]. In terms of the effective NLS equation, the new term would read $i\beta u_{xx}$. However, to comply with the FDT, one would also have to modify the correlations in (2b), adding new terms proportional to a derivative of the δ function. We will not pursue this generalization in the present work.

Another extension is the case of parametric noise. As a particular case one can consider a quasi-one-dimensional ferromagnet in an external magnetic field [2], which is described by a perturbed SG model. Assuming a randomly fluctuating field, one can bring the corresponding NLS equation for the small-amplitude excitations to the following form [cf. Eq. (5)]:

$$iu_t + u_{xx} + 2|u|^2 u = -i\alpha u + f(x,t)e^{it}u, \tag{23}$$

where $f(x,t)$ is again the Gaussian random function. After some algebra, the FP equation for the amplitude distribution function can be found as follows [cf. Eq. (14)]:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \eta}[(-2\alpha\eta + \epsilon^2\eta^2)P] + \frac{\partial^2}{\partial \eta^2}(\tfrac{2}{3}\epsilon^2\eta^3 P). \tag{24}$$

A crucial difference from Eq. (14) is that this time the effective diffusion coefficient is $\sim \eta^3$. The attempt to find a static distribution for Eq. (24) leads to the expression [cf. Eq. (20)]

$$\tilde{\Pi}(\eta) = \text{const} \times \eta^{-3/2} e^{-3\alpha/\epsilon^2\eta}. \tag{25}$$

Actually this solution cannot have any physical meaning: the distribution function does not vanish at $\eta = \infty$, and it contains an irregular singularity at $\eta = 0$. However, the solution in (25) can be regarded as a hint of the fact that parametric noise, unlike additive noise, cannot support any distribution of envelope solitons; the total probability collapses to the point $\eta = 0$. This parametric noise problem, which has been treated within stochastic perturbation for Gaussian random noise [11], requires further analysis.

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[1] F. G. Bass, Yu. S. Kivshar, V.V. Konotop, and Yu. A. Sinitsyn, Phys. Rep. **157**, 63 (1988).

[2] Yu. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. **61**, 763 (1989).

[3] S. E. Trullinger, M. D. Miller, R. A. Guyer, A.R. Bishop, F. Palmer, and J. A. Krumhansl, Phys. Rev. Lett. **40**, 206 (1978).

[4] M. Salerno, E. Joergensen, and M. R. Samuelsen, Phys. Rev. B **30**, 2 2635 (1984); D. J. Kaup and E. Osman, *ibid.* **33**, 1762 (1986); F. Marchesoni, *ibid.* **34**, 6536 (1986).

[5] B. A. Malomed, Physica D **29**, 113 (1987).

[6] D. J. Kaup and A. C. Newell, Phys. Rev. B **18**, 5162 (1978).

[7] H. Haken, *Synergetics. An Introduction*, 3rd ed. (Springer-Verlag, Berlin, 1983).

[8] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1964).

[9] G. Agrawal, *Nonlinear Fiber Optics* (Academic, San Diego, 1989).

[10] O. H. Olsen and M. R. Samuelsen, Phys. Rev. B **29**, 2803 (1984).

[11] F. G. Bass, Yu. S. Kivshar, V. V. Konotop, and G. M. Pritula, Phys. Lett. **70**, 309 (1989).

[12] R. Graham, Phys. Rev. A **25**, 3234 (1982).

[13] F. Marchesoni and C. Lucheroni, in *Condensed Matter Theories*, edited by A. N. Proto and J. L. Aliaga (Plenum, New York, 1992), Vol. 7, p. 351.